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LETTER TO THE EDITOR

Cartan's method of equivalence and second-order equation fields

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Abstract. The theory underlying Cartan's method of equivalence is outlined and applied to second-order equation fields. In particular, the technique is used to obtain the canonical horizontal distribution associated to a second-order equation field—a construction which has been presented before but in a rather *ad hoc* manner.

Some time ago, M Crampin, G E Prince and I investigated the inverse problem of the calculus of variations in a geometric context sufficiently comprehensive to allow for time-dependent systems (Crampin *et al* 1984). Indeed, our main objective was to show how certain results of Crampin, valid for time-independent systems, could be suitably modified to allow for explicitly time-dependent systems (Crampin 1981, 1983). We viewed a second-order equation field as a vector field Γ on the space $J^1(\mathbb{R}, M)$ (evolution space) which can be identified in a natural manner with $\mathbb{R}\times TM$; here, M (configuration space) is assumed to be a smooth m manifold. We showed that Γ could be used to define an m-dimensional 'horizontal' distribution on $J^1(\mathbb{R}, M)$ complementary to the vertical distribution obtained by viewing $J^1(\mathbb{R}, M)$ as a vector bundle over $\mathbb{R}\times M$. Unfortunately, we had to introduce the horizontal distribution (I shall refer to it here as $\mathscr{H}$) in a rather *ad hoc* manner, though much of our paper argued that the choice of $\mathscr{H}$ and the vector fields used to span it, was an auspicious one.

In this letter I wish to demonstrate how Cartan's method of equivalence leads in a natural manner to the discovery of the distribution $\mathscr{H}$. To render the sequel intelligible to the general reader, I shall briefly sketch the method of equivalence or, as it has come to be known, the theory of G structures, particularly as it applies to second-order equation fields (see also Cartan 1908, Sternberg 1964 and Kobayashi 1972).

A G structure is by definition a reduction of the frame bundle $\mathscr{F}(N)$ of a smooth n-manifold N to a principal subbundle B_{G} with structure group G; G is a subgroup of $\mathrm{GL}(n,\mathbb{R})$. If G is a subgroup of H, which is in turn a subgroup of $\mathrm{GL}(n,\mathbb{R})$, a G structure B_{G} may always be extended to an H structure B_{H}. On the other hand, if H is a subgroup of G, there does not exist in general an H structure B_{H}. If such a B_{H} does exist, it is said to be a reduction of B_{G} to B_{H}. Usually when one is given a G structure, one would like to find the smallest H structure to which it is reducible. The extreme case is when G is the trivial group and this of course is nothing but a complete parallelism on N.

Cartan (1908) developed a technique for reducing the group of a G structure which has only been understood comparatively recently. I shall now outline this technique.

Firstly, B_G has a canonically defined $\mathbb{R}^n$-valued 1-form ω, say. ω is the restriction of a canonical form, which I shall also denote by ω, on $\mathcal{F}(N)$, and is defined as follows: let $\pi\colon \mathcal{F}(N) \to N$ be the natural submersion, $p \in \mathcal{F}(N)$ with $\pi(p) = x$, $X \in T_p\mathcal{F}(N)$; then $\omega(X)_p$ is defined to be the element of $\mathbb{R}^n$ obtained by expanding $\pi_* X$ with respect to the frame p. Suppose now that f is a diffeomorphism of N such that the induced diffeomorphism $\tilde{f}$ of $\mathcal{F}(N)$ has the property that for any two frames p_1, p_2 at x differing by an element of G, $\tilde{f}(p_1)$ and $\tilde{f}(p_2)$ differ by an (in fact the same) element at G at $f(x)$; then $\tilde{f}$ induces by restriction a diffeomorphism, also denoted by $\tilde{f}$, of B_G fibred over f such that $\tilde{f}^*\omega = \omega$. Conversely, a diffeomorphism F of B_G fibred over a diffeomorphism f of M such that $F^*\omega = \omega$ is obtained by lifting f to B_G in just the same way.

Let the ith component of ω be denoted by ω^i and consider the $\mathbb{R}^n$-valued 2-form $d\omega$. Now, since the vanishing of the ω^i's corresponds to the vertical distribution, by Frobenius' theorem there must be a collection of 1-forms θ^i_j such that

$$d\omega^i = \theta^i_j \wedge \omega^j. \tag{1}$$

Sternberg (1964) shows that the θ^i_j's can be chosen so that they satisfy the Lie algebra relations of the canonical Maurer-Cartan form on G modulo the ω^i's. Indeed, any set of θ^i_j's satisfying equation (1) has this property. Now if $r = \dim G$, let π^α $(\alpha = 1, r)$ determine a distribution on B_G which is everywhere transverse to the vertical. Then we may write

$$d\omega^i = a^i_{\alpha j}\pi^\alpha \wedge \omega^j + T^i_{jk}\omega^j \wedge \omega^k \tag{2}$$

for some functions $a^i_{\alpha j}$ and T^i_{jk} on B_G. The $a^i_{\alpha j}$'s, T^i_{jk}'s and π^α's in equation (2) are in general not unique, but neither are they arbitrary. A key part of Cartan's reduction technique lies in modifying these elements of equation (2) in such a way, roughly speaking, as to simplify the second term on the right-hand side of equation (2), which is known as the torsion, whilst at the same time preserving the Lie algebra relations satisfied by the $a^i_{\alpha j}\pi^\alpha$'s. This technique is known as Lie algebra-compatible absorption of torsion.

The basic theory underlying group reduction is described in detail by Sternberg (1964). Briefly it goes as follows. As I mentioned above, the torsion term in equation (1) is not unique; suppose, however, we fix a point $p \in B_G$ and choose a subspace H of T_pB_G complementary to the vertical subspace. Then $d\omega|_H$, that is, $d\omega$ restricted to H, defines an element of $\Lambda^2 V^* \otimes V$, where for clarity I have put $\mathbb{R}^n = V$. Now G acts on B_G of course, but also on $\Lambda^2 V^* \otimes V$ in a natural way. Consider next the vector space $V^* \otimes V^* \otimes V$. Then, since the Lie algebra g of G is a subalgebra of $V^* \otimes V$ (g consists of endomorphisms of V) we have that $V^* \otimes \mathrm{g}$ is a subspace of $V^* \otimes V^* \otimes V$. Now define $\delta\colon V^* \otimes V^* \otimes V \to \Lambda^2 V^* \otimes V$ to be the canonical skew-symmetrising map. It is a routine calculation to show that for $w \in V^* \otimes \mathrm{g}$, $S \in G$

$$(\Lambda^2 \rho^\dagger \otimes \rho)(S)(\delta(w)) = \delta(\rho^\dagger \otimes \mathrm{Ad}(S)(w)), \tag{3}$$

where ρ denotes the action of G on V, $\rho^\dagger$ the contragredient action on V^* and Ad the adjoint representation of G. It follows from equation (3), amongst other things, that $\delta(V^* \otimes \mathrm{g})$ is an invariant subspace of $\Lambda^2 V^* \otimes V$ under the action of G. Hence there is a well defined action of G on $\Lambda^2 V^* \otimes V / \delta(V^* \otimes \mathrm{g})$. Sternberg (1964) shows that the image of $d\omega|_H$ in this quotient space is independent of H and hence there is a well defined map $C\colon B_G \to \Lambda^2 V^* \otimes V / \delta(V^* \otimes \mathrm{g})$.

Moreover, c satisfies for $S \in \mathrm{G}$, $p \in B_{\mathrm{G}}$

$$c(Sp) = S(c(p)). \tag{4}$$

It follows from equation (4) that $c(B_{\mathrm{G}})$ is a union of orbits of the action of G on $\Lambda^2 V^* \otimes V/\delta(V^* \otimes \mathrm{g})$. Let us assume that $c(B_{\mathrm{G}})$ simply consists of a single orbit; such a G structure is said to be of constant type and most known examples seem to be of this kind. Then as Sternberg shows, one may select any vector v in $c(B_{\mathrm{G}})$ and $c^{-1}(v)$ will be a reduction of the G structure to an H structure, where H is the isotropy group of v. In practice it is rather awkward to work on the quotient space $\Lambda^2 V^* \otimes V/\delta(V^* \otimes \mathrm{g})$; instead, it is more convenient to use the technique of Lie algebra-compatible absorption to choose a complement to $\delta(V^* \otimes \mathrm{g})$ in $\Lambda^2 V^* \otimes V$. The reduction then proceeds essentially as before.

It may happen that it is necessary to perform several reductions of a G structure B_{G} before one obtains a reduction to an H structure with H as small as possible. When this is finally achieved it is then necessary, if H is not trivial, to embark on the other main procedure in the equivalence method, namely, prolongation. In this, one considers a new equivalence problem on a G_1 structure whose base is B_{H}. However, I shall not discuss the technique of prolongation further in this letter.

I shall now show how the foregoing theory can be applied to second-order equation fields. I shall use local coordinates (t, x^i, u^i) on $J^1(\mathbb{R}, M)$ where t is the canonical coordinate on $\mathbb{R}$, (x^i) coordinates on M and (u^i) velocity coordinates for the fibres of $J^1(\mathbb{R}, M)$ over $\mathbb{R} \times M$. To employ the technique of equivalence it is first necessary to decide which kind of coordinate transformations are to be allowed. In Crampin *et al* (1984), we introduced a $1-1$ tensor called S, the local expression for which is $(\partial/\partial u^i) \otimes (\mathrm{d}x^i - u^i\,\mathrm{d}t)$. We attempted to show that S is an essential geometrical ingredient of $J^1(\mathbb{R}, M)$ and it played a key role in our analysis. Accordingly, it would therefore seem natural to consider coordinate transformations which at least preserve S.

Proposition. A diffeomorphism Φ preserves S iff Φ is the lift of a diffeomorphism ϕ of $\mathbb{R} \times M$ of the form $t' = \phi(t)$, $x^{i'} = \phi^i(t, x^j)$.

Proof. Notice that if S is considered as a linear transformation in each tangent space of $J^1(\mathbb{R}, M)$ its image consists of the vertical subspace. It follows that if Φ preserves S, it preserves the fibration of $J^1(\mathbb{R}, M)$ over $\mathbb{R} \times M$ so that Φ is fibred over a diffeomorphism ϕ of $\mathbb{R} \times M$. Let ϕ be given locally by $t' = \phi(t, x^j)$, $x^{i'} = \phi^i(t, x^j)$. Then it is easy to see by a calculation that ϕ is given by

$$u^{j'} = \frac{\partial\phi^j/\partial t + u^i(\partial\phi^j/\partial x^i)}{\partial\phi/\partial t + u^i(\partial\phi/\partial x^i)}$$

together with the equations for t' and $x^{i'}$. Since the Jacobian of ϕ cannot vanish, it follows that the numerator and denominator in the expression for $u^{j'}$ cannot both vanish. However, the denominator cannot vanish, which forces $\partial\phi/\partial t \neq 0$ and $\partial\phi/\partial x^i = 0$, and Φ is of the form claimed.

Another point to note from the form of Φ given in the last proposition is that $\partial u^{j'}/\partial u^i = \partial\phi^j/\partial x^i$—an observation which will be of use in achieving an initial group reduction.

It would appear that the largest (pseudo-)group of coordinate transformations worth considering are those of the type described in the proposition just proved. I shall choose a subgroup A of these transformations, namely those for which $t' = t + a$, where a is constant. This choice is motivated by classical mechanics where one usually considers time as being determined up to a choice of initial starting point. One might also consider the still smaller subgroup in which $\partial\phi^i/\partial t = 0$; however, any invariants of A are invariants of this smaller group. Generally one expects calculations to be harder the smaller the group, precisely because, since it is harder for two G structures to be equivalent, there are more invariants. Notice that the term 'group' is being used here in two distinct senses. To return briefly to the general picture, there is the group (or pseudo-group) or coordinate transformations as well as the group G which is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. The two are related in that the collection of Jacobians of all coordinate transformations at a given point of N forms a subgroup of $\mathrm{GL}(n, \mathbb{R})$ which contains G as a subgroup. G will actually be a proper subgroup if it is required to preserve some more first-order geometric structure. This happens in G structures determined by second-order equation fields as I shall now explain.

Suppose that the second-order equation field Γ is given by the equations $\ddot{x}^i = f^i(t, x^j, \dot{x}^j)$. We may view this as the vector field $\partial/\partial t + u^i(\partial/\partial x^i) + f^i(\partial/\partial u^i)$ on $J^1(\mathbb{R}, M)$. Next, choose a coframe on $J^1(\mathbb{R}, M)$ which is to be adapted, as far as is possible, to the geometry of the problem. An obvious choice is $(\omega, \theta^i, \pi^i)$ where $\omega = \mathrm{d}t$, $\theta^i = \mathrm{d}x^i - u^i\,\mathrm{d}t$ and $\pi^i = \mathrm{d}u^i - f^i\,\mathrm{d}t$. This is a coframe which has been used frequently before (cf Crampin 1977, Prince 1983). Notice that Γ is related to this coframe by the relations

$$\langle\Gamma, \omega\rangle = 1, \qquad \langle\Gamma, \theta^i\rangle = 0, \qquad \langle\Gamma, \pi^i\rangle = 0. \tag{5}$$

The group G is now obtained by requiring that it be consistent with the group A but also that Γ be preserved, that is, one requires diffeomorphisms Φ of $J^1(\mathbb{R}, M)$ such that $\Phi_*\Gamma = \Gamma$. The following equation gives the form of the group G where $A_i^j \in \mathrm{GL}(m, \mathbb{R})$ and B_i^j is an arbitrary $m \times m$ matrix:

$$\begin{pmatrix} \omega \\ \theta^j \\ \pi^j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_i^j & 0 \\ 0 & B_i^j & A_i^j \end{pmatrix} \begin{pmatrix} \omega \\ \theta^i \\ \pi^i \end{pmatrix}. \tag{6}$$

Equation (6) expresses the most general possible change that can be undergone by the coframe $(\omega, \theta^i, \pi^i)$ if it is required to be consistent with A. Notice that in view of the remark following the proposition, namely that $\partial u^{j'}/\partial u^i = \partial\phi^j/\partial x^i$, the lower $m \times m$ block has to be identical to the central $m \times m$ block. One should always see if it is possible to make initial group reductions in this manner. (Actually, even if one did not impose this condition at the outset, it turns out in this case that one obtains it by applying the general procedure outlined above.)

Thus far, equation (6) has been interpreted in terms of the geometry of $J^1(\mathbb{R}, M)$; however, it can also be interpreted in terms of a G structure with base $J^1(\mathbb{R}, M)$ and group consisting of $(2m+1) \times (2m+1)$ matrices of the form appearing in equation (6). In this interpretation, the entries in the matrices A_i^j and B_i^j are fibre coordinates. Moreover, the forms $(\omega, \theta^i, \pi^i)$ are a local representative of the canonical $\mathbb{R}^{2m+1}$-valued form on the G structure. Using equation (6), I now compute the derivatives of $(\omega, \theta^i, \pi^i)$ in the manner of equation (1). Notice that the Maurer–Cartan forms are given by

$(2m+1)\times(2m+1)$ matrices of the form (a^j_i, and b^i_j being arbitrary $m\times m$ matrices)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a^i_j & 0 \\ 0 & b^i_j & a^i_j \end{pmatrix}.$$

Thus one obtains

$$\begin{pmatrix} \mathrm{d}\omega \\ \mathrm{d}\theta^i \\ \mathrm{d}\pi^i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^i_j & 0 \\ 0 & b^i_j & a^i_j \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \theta^j \\ \pi^j \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda^i_j\omega\wedge\pi^j + \mu^i_{jk}\pi^j\wedge\pi^k + \nu^i_{jk}\theta^j\wedge\theta^k \\ \rho^i_{jk}\pi^j\wedge\pi^k + \sigma^i_j\omega\wedge\pi^j \end{pmatrix}. \tag{7}$$

In equation (7) the Lie algebra-compatible absorption step has been performed. There is a zero in the torsion term corresponding to $\mathrm{d}\omega$ because $\omega=\mathrm{d}t$ implies $\mathrm{d}\omega=0$. In the torsion corresponding to $\mathrm{d}\pi^i$, any terms involving θ^j can be absorbed into the non-torsion part because of the b^i_j block leaving the terms in $\pi^j\wedge\pi^k$ and $\omega\wedge\pi^j$. Finally, in the torsion term corresponding to $\mathrm{d}\theta^i$, terms in $\omega\wedge\theta^j$ and $\theta^j\wedge\pi^k$ can be absorbed in such a way as to preserve that Lie algebra relations by modifying the coefficients ρ^i_{jk} and σ^i_j. Equation (7) exhibits the maximal absorption which can be effected using only the form of the Lie algebra relations and the fact that $\mathrm{d}\omega=0$. In other words, we have not used the specific expressions for the forms θ^j and π^j in (6) but only our knowledge of form of the group and hence Lie algebra.

The next step consists of calculating the torsion terms using (ω, θ^j, π^j) and back-substituting from equation (6). This process is sometimes referred to as the 'parametric' calculation of torsion, as opposed to the previous step which is said to be 'intrinsic', in as much as it is independent of any local trivialisation, that is, choice of coordinates. In fact when we compute parametrically and mimic the intrinsic absorptions, which is absolutely essential, we obtain

$$\begin{pmatrix} \mathrm{d}\omega \\ \mathrm{d}\theta^i \\ \mathrm{d}\pi^i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^i_j & 0 \\ 0 & b^i_j & a^i_j \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \theta^j \\ \pi^j \end{pmatrix} + \begin{pmatrix} 0 \\ \omega\wedge\pi^i \\ (A^{-1})^l_k(2B^j_l + A^j_1\partial f^i/\partial u^l)\omega\wedge\pi^k \end{pmatrix}. \tag{8}$$

Moreover, a^i_j and b^i_j are given explicitly by the formulae

$$a^i_j = \mathrm{d}A^i_k(A^{-1})^k_j - B^i_k(A^{-1})^k_j \tag{9a}$$

$$b^i_j = (A^{-1})^k_j\,\mathrm{d}B^i_k - (A^{-1})^k_j(A^{-1})^m_l B^l_k\,\mathrm{d}A^i_m + [(A^{-1})^k_j A^i_j\partial\Gamma^l/\partial x^k - (A^{-1})^k_j(A^{-1})^m_l B^l_k(B^i_m + A^i_n\partial\Gamma^n/\partial u^m)]\omega. \tag{9b}$$

In practice, one is not especially interested in the explicit forms of quantities such as those given by equations (9*a*) and (9*b*), though these will be obtained naturally in the course of the calculation. On the other hand, one is very much interested in the coefficients of the torsion, for it is these which allow the possibility of group reduction. In equation (8), the torsion coefficients corresponding to $\mathrm{d}\omega$ and $\mathrm{d}\theta^i$ are constant and so are of no use as regards the reduction technique. However, the torsion corresponding to $\mathrm{d}\pi^i$ is not constant and one must determine how it transforms under the action of G. To do this, we can determine the differential of the action from equation (8) and thus the action itself.

From the fact that $d^2\pi^i = 0$ one obtains from equation (7), using the fact that $\rho^i_{jk} \equiv 0$ and $d\omega = 0$,

$$db^i_j \wedge \theta^j - b^i_j \wedge d\theta^j + da^i_j \wedge \pi^j - a^i_j \wedge d\pi^j + d\sigma^i_j \wedge \omega \wedge \pi^j - \sigma^i_j \omega \wedge d\pi^j = 0. \tag{10}$$

The term involving $d\sigma^i_j$ in equation (10) includes ω and π^j, so we must wedge in a complementary set of the canonical 1-forms, say, $\omega^2 \wedge \ldots \wedge \omega^m \wedge \theta^1 \wedge \ldots \wedge \theta^m$, so that only the value $j = 1$ in the term $d\sigma^i_j \wedge \omega \wedge \pi^j$ will be significant. Equation (7) can be used in the resulting equations to eliminate the terms in $d\theta^j$ and $d\pi^j$. (Alternatively, the last two steps can be performed in the reverse order. In the current problem the order indicated seems easier.) One obtains eventually

$$(d\sigma^i_k - b^i_k + \sigma^i_j a^j_k - \sigma^j_k a^i_j) \wedge \omega \wedge \pi \wedge \theta + (da^i_k - a^i_j \wedge a^j_k) \wedge \pi \wedge \theta = 0 \tag{11}$$

where

$$\pi = \pi^1 \wedge \ldots \wedge \pi^m \qquad \text{and} \qquad \theta = \theta^1 \wedge \ldots \wedge \theta^m.$$

From equation (11) it follows by wedging with ω that the forms $da^i_k - a^i_j \wedge a^j_k$ belong to the algebraic ideal generated by ω, π^i and θ^i. Hence we may conclude that

$$d\sigma^i_k + \sigma^i_j a^j_k - \sigma^j_k a^i_j - b^i_k \equiv 0 \bmod(\omega, \theta^l, \pi^l). \tag{12}$$

Equation (12) implies that the action of G on the components of the torsion in (7) corresponding to $d\pi^i$ consists of conjugation and translation. Accordingly, we may 'translate σ^i_k to zero', that is, choose a vector in the complement to $\delta(V^* \otimes g)$ for which the σ^i_k are zero. We know the 'parametric' form of σ^i_k from equation (8) and we see that we have imposed the conditions

$$B^j_l = -\tfrac{1}{2} A^j_i \partial f^i / \partial u^l. \tag{13}$$

If we compare this to equation (6) we can interpret equation (13) as follows: we have reduced our original G structure to a $\mathrm{GL}(m, \mathbb{R}) \times \mathrm{GL}(m, \mathbb{R}) \times \{1\}$-structure and modified the original choice of $\boldsymbol{\omega}$, $\boldsymbol{\theta}^i$ and $\boldsymbol{\pi}^i$. In fact the modified coframe of $J^1(\mathbb{R}, M)$ is given by

$$\boldsymbol{\omega} = dt, \qquad \boldsymbol{\theta} = dx^i - u^i\, dt$$

and

$$\boldsymbol{\pi}^i = du^i - f^i\, dt - \tfrac{1}{2}(\partial f^i/\partial u^l)(dx^l - u^l\, dt).$$

We thus pass to a new equivalence problem in which the analogue of equation (6) is

$$\begin{pmatrix} \omega \\ \theta^j \\ \pi^j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A^j_i & 0 \\ 0 & 0 & A^j_i \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{\theta}^i \\ \boldsymbol{\pi}^i \end{pmatrix}. \tag{14}$$

It remains only to observe that the modified forms $(\boldsymbol{\omega}, \boldsymbol{\theta}^i, \boldsymbol{\pi}^i)$ are precisely the same forms that in Crampin *et al* (1984) were denoted by (dt, θ^i, ψ^i) and that the horizontal distribution $\mathscr{H}$ is determined by the vanishing of the forms $\boldsymbol{\omega}$ and $\boldsymbol{\pi}^i$.

The equivalence method can be continued to be applied to the new problem with reduced structure group. Naturally, as the process continues, the calculations become more cumbersome. One successively obtains more redefined coframes which depend on higher and higher derivatives of the 'data'—in our case the f^i's. I shall not give the details here as, amongst other things, they involve the technique of prolongation.

Suffice it to say, that the equivalence of second-order equation fields is one of a variety of geometric problems in mathematical physics to which the method of equivalence should be applicable.

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